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Brownian Motion in Curved Spacetimes

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Abstract

In this report the effects of a curved spacetime geometry on Brownian motion are explored. A recent paper by Smerlak, in which a curved-spacetime diffusion equation is derived, is discussed together with supporting background material. A calculation from this paper, which relates to the Schwarzschild constant density star model, is corrected. This correction shows that curvature effects can significantly affect the motion of a Brownian particle in astrophysical systems, contrary to what was found in the original paper. We then extend this analysis to static stellar models that are more physically realistic, namely the Tolman IV model and the Goldman III model. Lastly, we derive the small-time asymptotic expansion for the mean squared displacement of a Brownian particle in the Friedmann-Lemaître-Robertson-Walker cosmological models. Specific models are then considered to investigate the effects of expansion on Brownian motion.

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Chapter 1

Introduction

1.1 Diffusion and Brownian motion

Diffusive processes are ubiquitous in nature. Familiar examples of diffusion include the spreading of a gas into a room and the movement of a coloured dye in a beaker of water. Roughly speaking, diffusion is any process whereby particles spread in some medium, independent of the bulk motion of the medium. It turns out that many phenomena are well-modelled mathematically by diffusion; for example, diffusive phenomena underlie not only thermal conduction and osmosis, but also the spreading of price values in an economic context and opinion spreading amongst a crowd of people.

Some of the simplest diffusion processes, which are also those that are most important for this report, are those that fall into the category called *Brownian motions*. In physics, Brownian motion refers to the observed irregular motion of tiny particles that are suspended in a fluid. This was originally noticed by Brown in 1827, when he used a microscope to observe pollen grains floating in water. In 1905, Einstein's *Annus Mirabilis*, Einstein explained this motion as being due to the continual bombardment of the Brownian particle by the relatively small molecules of the surrounding water [1]. Due to the large number of fluid particles, which hit the Brownian particle from many different directions, this process is described *stochastically*, which means that it is a fundamentally probabilistic process rather than a deterministic one. Specifically, in the jargon of stochastic modelers, Brownian motion is mathematically described as a "continuous-time, continuous-state-space Markov process" [2][1]. Since this is a stochastic process, a Brownian particle's motion cannot be predicted exactly, but instead its position is described using a probability density function p. The probability density function describes the probability of finding the particle in a given region after a given time. Using physical arguments, Einstein showed that, for a one-dimensional Brownian motion, p satisfies the partial differential equation

$$\frac{\partial p}{\partial t} = \kappa \frac{\partial^2 p}{\partial x^2}. \tag{1.1}$$

The constant κ, which has units $m^2 s^{-1}$, is called the *diffusivity* and it measures the rate at which the Brownian particle spreads, which will be shown in section 2.2. Although initially

[1]Another way of stochastically describing Brownian motion is as a so-called *Wiener process*. Figure 1.1 shows the path of a one-dimensional Brownian particle generated by modelling Brownian motion as a Wiener process.

used to refer just to the irregular motion of tiny particles in a fluid, the term Brownian motion now encompasses a large range of phenomena occurring in many fields, all of which are mathematically described in the same way as this original physical process. For example, Brownian motion phenomenon occur in biophysics, quantum gravity, colloidal physics, and financial markets.

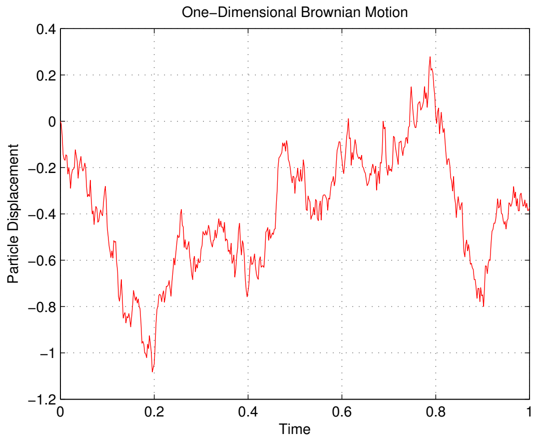


Figure 1.1: *A sample path for a one-dimensional Brownian particle generated using a Wiener process.*

1.2 Relativity

In addition to explaining the Brownian motion of small particles, another of Einstein's great achievements in 1905 was formulating the special theory of relativity. This theory, which showed how to relate measurements made by different observers in the absence of a gravitational field, demonstrated that light must have a finite speed in order to preserve causality. The special theory of relativity was the forerunner to the general theory of relativity, which Einstein had fully published by 1916. The general theory of relativity showed that the Newtonian notion of a gravitational *force* must be replaced by a geometric description of gravity in terms of the *curvature* of spacetime.

Following the acceptance of Einstein's relativity theories, an obvious task left for physicists was to develop relativistic formulations of the classical theories, those that were established before 1905. An interesting example of this is the attempts to develop a theory of "Relativistic Thermodynamics", including concerted efforts by Einstein himself. In this case, "relativistic" refers to special relativity; that is, Relativistic Thermodynamics is a theory of thermodynamics that attempts to incorporate special relativistic causality. The long and confused history of Relativistic Thermodynamics[2] serves to illustrate the difficulty in formulating relativistic versions of classical theories. This history also suggests that progress often necessitates a reassessment of what is understood about the most basic physical concepts, such as, in this case, temperature and entropy.

[2]See [3] for an interesting discussion of this.

There have also been related attempts to formulate a *relativistic theory of dissipative hydrodynamics*. Dissipative hydrodynamics is the study of dynamical fluids that have heat-generating internal processes; in particular, such theories describe heat conduction. The first relativistic theory of dissipative hydrodynamics originated with Eckart [4] in 1940. Eckart wrote down the simplest general relativistic generalisation of the classical heat equation that is consistent with the second law of thermodynamics. Although this theory can be written covariantly, and is thus compatible with general relativity, it turns out that it allows dissipative perturbations to propagate at infinite speeds. Thus, Eckart's theory does not preserve special-relativistic causality, which led some physicists to abandon it. However, as argued in [5], the presence of infinite speeds is not fatal to the theory; it is merely a reflection of the fact that Eckart's theory has a certain domain of validity. Essentially, the Eckart theory is a macroscopic description that averages over the microscopic physics and that breaks down when spacetime gradients arise on scales comparable to the length and time scales of the dissipative processes.

1.3 Relativistic diffusion

This report sits within the broader context of efforts to formulate a general relativistic theory of diffusion, and a fortiori Brownian motion. In particular, this report aims to understand the effects of a *curved spacetime geometry* on the motion of a Brownian particle. Thus, *no attempt is made to include special relativistic causality in the framework of this report.* Put another way, the theory presented here is the limit of a fully general relativistic theory of diffusion in which the speed of light becomes infinite.

General relativistic effects become important whenever spacetime curvature cannot be neglected—when gravity is strong. Thus, when one wants to describe Brownian motion, or any diffusive process, in the presence of a non-negligible spacetime curvature, a general relativistic description becomes desirable. Such a situation arises, for example, when considering the diffusive component of the fluid motion within a massive star. The other context in which general relativity cannot be neglected is the early universe. In the early universe, general relativistic effects were strong enough to affect diffusive processes, such as photon and neutron diffusion. Lastly, there are also applications of general relativistic diffusion to so-called *analogue spacetimes*—systems that can be described by an effective curved spacetime, see for example [6] and [7].

As mentioned above, this report considers the $c \to \infty$ limit of a fully relativistic theory of Brownian motion, where c is the speed of light. The different problem of developing a *causal* theory of Brownian motion has been well-studied. To construct a causal theory, one attempts to develop a phenomenological description of special relativistic diffusion processes by unifying the underlying relativistic and stochastic concepts. Since the regime of this report is somewhat different, the interested reader is directed to the recent review article [8] for more details on causal theories of Brownian motion. Given the difficulty in formulating a special-relativistic theory of Brownian motion, one can hope to learn something about a fully general relativistic theory[3] by studying the effects of spacetime curvature on a Brownian

[3] Debbasch claims to have a fully general relativistic theory for a particular form of diffusion called the

particle without a bound on the speed of light. That is where this work fits in.

1.4 Outline of research

This report focuses on a recent paper by Smerlak [10] in which a curved-spacetime diffusion equation is derived and discussed. An error was found in Smerlak's reasoning pertaining to a calculation of the mean squared displacement (MSD) of a Brownian particle in a Schwarzschild constant density star. Once this calculation is corrected, it is found that curvature effects can have a significant effect on the motion of a Brownian particle within the star, contrary to the result obtained by Smerlak. Specifically, in dense enough stars, the curvature correction to the MSD can make a Brownian particle spread much faster on average than it would in a Euclidean space. Similar calculations are performed using other stellar models that are more physically realistic than the Schwarzschild constant density star model. These stellar models are physically realistic static spherically symmetric perfect fluid solutions of Einstein's field equations. In these models, namely the Tolman IV model and Goldman III model, spacetime curvature is found to have a significant effect on the motion of a Brownian particle for at least some values of the model parameters. Moreover, the effect of curvature in all of the models considered is to make a Brownian particle spread faster than it would in a Euclidean space.

The small-time asymptotic expansion for the mean squared displacement of a Brownian particle in the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models is derived. Since these models are non-static, this involves a significant generalisation of the calculations contained in Smerlak's paper [10]. The result is then explored by considering Brownian motion in an Einstein-de Sitter universe and a de Sitter universe.

Ornstein-Uhlenbeck process, eg. [9].

Chapter 2

Diffusion in curved spacetimes

In this chapter the necessary background and notation needed to derive the curved-spacetime diffusion equation is established. The first section introduces the 3+1 splitting of spacetime. The second section discusses classical diffusion as a stochastic process. In the third and final section, the curved-spacetime diffusion equation is obtained.

2.1 The 3 +1 decomposition and Einstein's equations

Let $\mathcal{M}$ be a 4-dimensional spacetime manifold with a Lorentzian metric g of signature $(-,+,+,+)$ and use natural units where the speed of light c is set equal to one. The standard requirement that our spacetime is *globally hyperbolic* is assumed—this means that $(\mathcal{M}, g)$ admits a *Cauchy surface* Σ, a spacelike hypersurface such that each causal curve intersects Σ exactly once. In other words, specifying initial conditions on a Cauchy surface determines the future and past uniquely[1]. A globally hyperbolic spacetime $\mathcal{M}$ can be foliated by a family of spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$ that are parameterised by a global time coordinate t. Each of these hypersurfaces is a constant time level surface of t,

$$\Sigma_{t'} = \{p \in \mathcal{M} : t(p) = t'\},$$

and the three-dimensional spacelike hypersurfaces Σ_t cover $\mathcal{M}$.

Consider now a relativistic fluid with velocity u^μ; this fluid will be the substrate for the diffusion processes to be described later. The four-vector u^μ also defines the velocity of the so-called *fiducial* observers, which are observers that are at rest with respect to, or *comoving* with, the cosmic fluid. It is assumed that the fluid flow is irrotational, that is, $u_{[\mu}\nabla_\nu u_{\alpha]} = 0$, where ∇_ν is the covariant derivative with the Levi-Civita connection on $\mathcal{M}$. Then, by the Frobenius theorem, there exists a foliation of $\mathcal{M}$ by hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$ which are orthogonal to u^μ. That is, the worldlines of the fiducial observer are orthogonal to the hypersurfaces Σ_t. The hypersurfaces Σ_t thus represent locally the set of events that are simultaneous from the point of view of a fiducial observer. In the rest-frame of a fiducial

[1]In astrophysical and cosmological problems, spacetimes are usually chosen to be globally hyperbolic for simplicity.

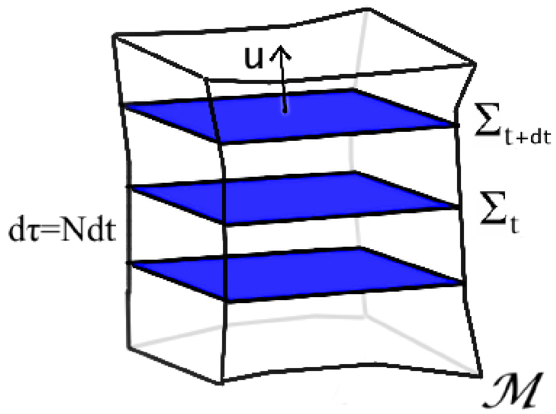


Figure 2.1: *The 3+1 foliation of spacetime.*

observer, the metric can be written as[2]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + h_{ij}dx^i dx^j,$$

where $N = N(t, x^k)$ is called the lapse function. In this set of coordinates

$$u_\mu = -N\nabla_\mu t = (-N, 0, 0, 0). \tag{2.1}$$

The lapse function N relates the coordinate time dt to the proper time $d\tau$ of a fiducial observer with the relation $d\tau = Ndt$. The quantity $h_{ij} = h_{ij}(t, x^k)$ is the intrinsic metric induced on the Riemannian[3] spatial hypersurfaces Σ_t by the full 4-dimensional metric, $g_{\mu\nu}$. The hypersurfaces Σ_t also inherit an associated covariant derivative D_i and the so-called Laplace-Beltrami operator, $\Delta_h = h^{ij}D_jD_i$. The Laplace-Beltrami operator is the generalisation to Riemannian manifolds of the usual Euclidean Laplacian operator and can be written in a coordinate basis as

$$\Delta_h(\cdot) = \frac{1}{\sqrt{|h|}}\partial_i\Big(\sqrt{|h|}h^{ij}\partial_j(\cdot)\Big), \tag{2.2}$$

where h denotes the determinant of the intrinsic metric and h^{ij} are the components of the inverse of the intrinsic metric. The Laplace-Beltrami operator will turn out to be important for our considerations of Brownian motion in curved spacetimes.

A quantity that is important when discussing expanding spacetimes is the expansion scalar $\theta = \nabla_\mu u^\mu$. It can be shown that θ measures the fractional rate of change of an infinitesimal volume δV about a spatial point along the fluid flow, viz.

$$\theta = \frac{1}{\delta V}\frac{d(\delta V)}{d\tau} = \frac{1}{N}\frac{1}{\delta V}\frac{d(\delta V)}{dt}, \tag{2.3}$$

which is shown explicitly in [11].

[2]Latin indices run from 1 to 3, Greek indices from 1 to 4, and the Einstein summation rule is assumed for repeated indices.

[3]A Riemannian manifold has a positive definite metric.

The above discussion pertains to geometry. In order to be able to describe the interaction between the spacetime geometry and the matter contained within it, one needs *Einstein's field equations*. These can be compactly written as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.4}$$

where $G_{\mu\nu}$ is the Einstein tensor, Λ the cosmological constant, G Newton's gravitational constant and $T_{\mu\nu}$ the stress-energy tensor. Later, specific solutions to (2.4) will form the theoretical astrophysical and cosmological backgrounds within which Brownian motion is studied.

2.2 Diffusion as a stochastic process

In this section diffusion is discussed. Firstly there is some preliminary discussion of Brownian motion. This is followed by an introduction to diffusion as a Markov process on a Riemannian manifold by considering the relevant master and Fokker-Planck equations. The discussion follows that given by Smerlak in [10] with extra explanatory detail given where needed to make the discussion more accessible. For further discussion of Brownian motion in Riemannian manifolds, see [12].

2.2.1 Classical Brownian motion

Consider a one-dimensional Brownian particle that is initially at a position x_i at a time t_i. Let $p_t(x, x_i)$ denote the probability distribution function of the particles displacement. That is, the probability of finding the particle between x_1 and x_2 at a time $t \geq t_i$ is given by

$$\int_{x_1}^{x_2} p_t(x, x_i)\, dx.$$

As mentioned in the introduction, $p_t(x, x_i)$ satisfies the partial differential equation

$$\frac{\partial p}{\partial t} = \kappa \frac{\partial^2 p}{\partial x^2}, \tag{2.5}$$

where κ is the diffusivity, with the initial condition

$$\lim_{t \to t_i} p_t(x, x_i) = \delta(x - x_i), \tag{2.6}$$

where δ is the Dirac delta distribution. This initial condition says that at the time t_i the particle will be found at the position x_i with probability one.

In the context of heat conduction, equation (2.5) describes the distribution of heat and is called *the heat equation*. Equation (2.5) is also sometimes called *the diffusion equation*, but from a modern viewpoint it is seen to be a specific instance of a large class of partial differential equations which model more general diffusive processes. Solving equation (2.5) with the initial condition (2.6) gives the so-called *heat kernel* solution:

$$p_t(x, x_i) = \frac{1}{\sqrt{4\pi\kappa(t - t_i)}} \exp\Big(- \frac{(x - x_i)^2}{4\kappa(t - t_i)}\Big), \tag{2.7}$$

which is the familiar Gaussian distribution.

Since the Brownian particle is equally likely to move in any direction, the expected value of its displacement is zero. However, one can characterize the spreading of a Brownian particle using the *mean squared displacement* (MSD). By definition, the MSD, which is denoted by $\langle d^2(x, x_i)\rangle_t$, is the expected value of the squared distance between the Brownian walker's position at time t and its initial position:

$$\langle d^2(x, x_i)\rangle_t = \int (x - x_i)^2 p_t(x, x_i)\, dx,$$

where the integral is over all of position space.[4] For the one-dimensional heat kernel solution (2.7), one finds that

$$\langle d^2(x, x_i)\rangle_t = 2\kappa(t - t_i).$$

A simple generalisation of this calculation gives the result that the MSD of a Brownian particle in a $D-$dimensional Euclidean space is $2D\kappa(t - t_i)$. This result demonstrates the *diffusion scaling law*: the spatial spreading of a local impulse grows like the *square root* of time, where the speed of the average spreading is governed by the diffusivity κ, as promised earlier.

The above discussion considered the probability distribution function of a single Brownian particle. However, it is perfectly valid to describe an ensemble of Brownian particles by simply reinterpreting the probability distribution function as a particle density function. The heat kernel solution (2.7) then describes the spreading of the ensemble density. In particular, (2.7) tells us that there is a *non-zero* particle density everywhere when $t > t_i$. In the single particle interpretation, there is a non-zero probability of finding our single particle at any spatial point. This implies that our particles move instantaneously—their velocities are infinite. Thus the classical theory of Brownian motion is acausal. The acausality can be traced back to the fact that equation (2.5) is a *parabolic* partial differential equation.

2.2.2 Stochastic diffusion

Let Σ be a Riemannian manifold with metric h_{ij} and covariant derivative D_i. Let $x_t \in \Sigma$ denote the instantaneous position of a *random walker* at time t—that is, a diffusing particle in Σ that travels on a path consisting of a sequence of random steps. It is assumed that the diffusing particle can be described by a *Markov process*. This means that the current position, x_t, of the random walker will completely determine its later positions x'_t for $t' > t$, according to *transition rates* $\Gamma(x_t \to x'_t)$. The transition rates are defined such that the probability for the random walker to jump from a volume $dV(x)$ about $x \in \Sigma$ to a volume $dV(x')$ about $x' \in \Sigma$ in a time dt is given by

$$\Gamma(x \to x')\, dV(x)\, dV(x')\, dt.$$

Let $p_t(x, x_i)$ denote the probability density function, which will often be shortened to $p_t(x)$. Then the probability flux for a walker to go from x to x' is defined by

$$j_t(x \to x') = p_t(x)\, \Gamma(x \to x'). \tag{2.8}$$

[4] In general, the expectation value of some scalar function $\Omega(x)$ over a manifold Σ is given by $\langle \Omega(x)\rangle_t = \int_\Sigma dV\, \Omega(x) p_t(x, x_i)$.

Since the incoming and outgoing probability fluxes must balance at a point, relation (2.8) can be used to write the time variation of the probability at x as

$$\int_\Sigma dV(x')\,\Big(j_t(x'\to x) - j_t(x\to x')\Big) = \int_\Sigma dV(x')\,\Big(p_t(x')\,\Gamma(x'\to x) - p_t(x)\,\Gamma(x\to x')\Big).$$

Since $\int_V dV(x)\,p_t(x,x_i)$ gives the probability of finding the particle in V at time t, another way of writing the time variation of the probability at a point x is

$$\lim_{V\to\{x\}} \frac{d}{dt}\int_V dV(x)\,p_t(x,x_i) = \lim_{V\to\{x\}}\int_V dV(x)\,\partial_t p_t(x,x_i) = \partial_t p_t(x,x_i). \tag{2.9}$$

Thus, the probability density function p_t satisfies the relation

$$\partial_t p_t(x) = \int_\Sigma dV(x')\,\Big(p_t(x')\,\Gamma(x'\to x) - p_t(x)\,\Gamma(x\to x')\Big). \tag{2.10}$$

In words, (2.10) says that the rate of change of the probability of finding our particle at a point x is given by the probability for our particle to arrive at x from some other point, minus the probability for our particle to leave x for some other point. Equation (2.10), which describes the time-evolution of our system, is known as the *master equation* for this process and is sometimes written as $\partial_t p_t = \mathcal{M} p_t$, where $\mathcal{M}$ is called the *master operator*.

The above discussion has assumed that our system can only be in a countable number of discrete states, which our particle jumps between. In the limit where the jumps become infinitely frequent and short-ranged, the transition rates Γ become distributional and the master operator $\mathcal{M}$ reduces to its second-order truncation $\mathcal{L}$ in a moment expansion[5],

$$\mathcal{L}p_t = -D_i(w_1^i p_t) + \frac{1}{2}D_iD_j(w_2^{ij}p_t), \tag{2.11}$$

where w_1^i is a vector field on Σ, called *the drift vector*, and w_2^{ij} is a rank-2 tensor field, called the *diffusion tensor*. Equation (2.11) is called a *Fokker-Planck* equation[6]. It describes the continuous time evolution of the probability distribution function. Stochastic processes that are described by the Fokker-Planck equation (2.11) are called *diffusion processes*. For more on the use of master equations and Fokker-Planck equations in the description of stochastic processes, the reader is referred to [13].

2.2.3 Brownian motion

The above discussion holds for any diffusive process defined by an appropriate drift vector and diffusion tensor. Brownian motion corresponds to the case for which $w_1^j = 0$ and $w_2^{ij} = 2\kappa h^{ij}$, where κ is again the diffusivity. The vanishing of the drift vector implies that there is no tendency for a Brownian particle to drift in a favoured direction. The form of the diffusion tensor ensures that the Laplace-Beltrami operator is recovered. For Brownian motion, the Fokker-Planck equation (2.11) thus becomes

$$\partial_t p_t = \mathcal{L}p_t = \frac{1}{2}D_iD_j(2\kappa h^{ij}p_t) = \kappa D_iD^ip_t = \kappa\Delta_h p_t, \tag{2.12}$$

[5] Specifically, this is the second order expansion of the Kramers-Moyal expansion. For mathematical details see [13] or [14].

[6] Master equations are the discrete analogues of Fokker-Planck equations.

where Δ_h is the Laplace-Beltrami operator, which is given by (2.2), and the metric preserving property, $D_j h^{ij} = 0$, has been used. For Brownian motion, the transition rates are given by

$$\Gamma(x' \to x) = \kappa \, \Delta_h \delta(x, x'),$$

where δ is the Dirac delta distribution on Σ and Δ_h acts on the x' variable. From this, the relevance of Δ_h to Brownian motion is evident.

Notice that (2.12) is clearly a generalised version of (2.5)—the Euclidean Laplacian is simply replaced with the Laplace-Beltrami operator. However, (cf., the discussion at the end section 2.2.1) this implies that the Brownian particles described by (2.12) move instantaneously. In this derivation, this acausalitly originates from the assumption that our process is Markov. Indeed, in spacetime there are no nontrivial relativistic Markov processes [8], which makes it difficult to formulate a special-relativistic theory of Brownian motion.

2.3 Curved-spacetime diffusion equation

In this section a *curved-spacetime diffusion equation* is found by generalising the derivation of section 2.2 to curved spacetimes. A full derivation is not given here; that can be found in [10]. Rather, an outline is given of the main steps needed to arrive at a curved-spacetime diffusion equation. The explanation is also more qualitative than that given by Smerlak.

The first step is to generalise the master equation (2.10) and the Fokker-Planck equation (2.11) for Markov processes to describe diffusive processes in a curved spacetime. As discussed at the end of 2.2.3, since the system is still treated as a Markov process, the resultant diffusion equation will be acausal.

2.3.1 The curved-spacetime master equation

The discussion in section 2.2 of a random walker on a Riemannian manifold is taken to define the instantaneous dynamics of a random walker in spacetime, in proper time. That is, given an irrotational flow u^μ, we consider the associated foliation of spacetime by hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$. Each hypersurface Σ_t has its own set of transition rates, defined in terms of the intrinsic metric of that surface. In general, these transition rates will vary between different surfaces since the intrinsic metrics change, for example, when space is expanding. It is then assumed that the probability for a random walker carried by the flow u^μ to jump from $x_t \in \Sigma_t$ to $x'_t \in \Sigma_t$ in proper time $d\tau(x_t)$ is given by

$$\Gamma_t(x_t \to x'_t) \, dV(x_t) \, dV(x'_t) \, d\tau(x_t).$$

An important difference between the classical and curved-spacetime processes is that in the latter there is no single preferred time coordinate. Rather, as Smerlak argues, the physics will be naturally written in terms of the proper time that is measured along the particle wordlines. However, the final probability equation, which is necessarily global, should be written in terms of the global time coordinate. Hence, the master equation will involve the lapse function N, since this converts proper times to coordinate times. In particular, the probability flux becomes

$$j_t(x_t \to x'_t) = N(x_t) p_t(x_t) \, \Gamma_t(x_t \to x'_t). \tag{2.13}$$

Equation (2.13) represents the intuitive fact that the random walker jumps more frequently (high j) whenever the proper time runs faster (high N). This implies that the right-hand side of the curved-spacetime master equation is

$$\int_{\Sigma_t} dV(x_t') \Big(N(x_t') p_t(x_t') \, \Gamma_t(x_t' \to x_t) - N(x_t) p_t(x) \, \Gamma_t(x_t \to x_t') \Big).$$

The left-hand side of the master equation (2.10) is also modified in a curved spacetime. This stems from the fact that an infinitesimal volume element about a spatial point in the hypersurface changes with time. That is, the analogue of equation (2.9) is nontrivial. Recall, however, that (2.3) gives the change of an infinitesimal volume along a flow line in terms of the expansion scalar θ. Thus one has

$$\begin{aligned} \lim_{V_t \to \{x_t\}} \frac{d}{dt} \int_{V_t} dV(x_t) \, p_t(x_t) &= \lim_{V_t \to \{x_t\}} \int_{V_t} dV(x_t) \left(\partial_t p_t(x_t) + \frac{1}{\delta V} \frac{d(\delta V)}{dt} p_t(x_t) \right) \\ &= \partial_t p_t(x_t) + N\theta p_t. \end{aligned}$$

The curved-spacetime master equation for a Markov process on Σ_t is thus given by

$$\partial_t p_t(x_t) + N\theta p_t = \int_{\Sigma_t} dV(x_t') \Big(N(x_t') p_t(x_t') \, \Gamma_t(x_t' \to x_t) - N(x_t) p_t(x) \, \Gamma_t(x_t \to x_t') \Big). \quad (2.14)$$

As expected, (2.14) conserves the total probability $\int_{\Sigma_t} dV(x_t) \, p_t(x_t)$.

2.3.2 Brownian motion in a curved spacetime

Assuming that the Markov process is of diffusive type and following the same steps from section 2.2, the curved-spacetime Fokker-Planck equation is given by

$$\partial_t p + N\theta p = -D_i(w_1^i N p) + \frac{1}{2} D_i D_j (w_2^{ij} N p).$$

Using the characterisation of Brownian motion given in subsection 2.2.3, that is, $w_1^j = 0$ and $w_2^{ij} = 2\kappa h^{ij}$, gives the *curved-spacetime Fokker-Planck equation* for Brownian motion:

$$\partial_t p + N\theta p = \kappa \Delta_h (Np). \quad (2.15)$$

2.3.3 The static limit and connection to Eckart's approach

The static limit of equation (2.15) will be important when considering Brownian motion in static star models. By definition, a static spacetime is one in which there exists a time coordinate t such that all metric components are independent of t and the geometry is unchanged under time reversal, $t \to -t$. From (2.3), it is clear that in such a spacetime the expansion scalar θ vanishes. Thus the static limit of (2.15) is

$$\partial_t p = \kappa \Delta_h (Np). \quad (2.16)$$

It turns out that one can recover Eckart's heat flux ansatz from (2.16), mentioned in section 1.2. This means that Smerlak's argument gives a stochastic justification for Eckart's phenomenological equation. Note also that in a static spacetime, the spatial hypersurfaces Σ_t are unchanging with time and hence can all be identified with a single hypersurface Σ.

2.3.4 Mean squared displacement in a static spacetime

In [10], Smerlak demonstrates how to calculate the MSD for a Brownian particle in a static spacetime. That calculation is generalised to non-static FLRW spacetimes in section 3.2 and thus is not repeated here. The result, in the case of a static spacetimes, is

$$\langle d^2(x, x_i)\rangle_t = \sum_{n=0}^{\infty} \frac{(\kappa t)^n}{n!} \big((N\Delta_h)^n d^2\big)(x_i, x_i), \tag{2.17}$$

where $x_i \in \Sigma$ is the initial position of the Brownian particle, d is the Riemannian distance on Σ and Δ_h is the Laplace-Beltrami operator on Σ. Note that d and Δ_h do not change with time in static spacetimes; this will not be the case in an expanding universe. Also, since in static spacetimes the initial time is arbitrary, one can set $t_i = 0$.

At this point Smerlak assumes “without loss of generality” that $N(x_i) = 1$ and expands (2.17) up to second order. We have found that this assumption is not valid when $N(x_i) \neq 1$ since the term $N(x_i)$ appears nontrivially in the curvature correction terms to the MSD. Consequently, Smerlak incorrectly carries over the assumption that $N(x_i) = 1$ to a calculation of the motion of a Brownian particle in a particular stellar model. This mistake is corrected in section 3.1, however, this requires a slightly more general expansion of (2.17). The result of this more general expansion up to second order, details of which are in appendix A, is

$$\langle d^2(x, x_i)\rangle_t = 6\kappa t N(x_i)\Big(1 + \kappa t N(x_i)\Big(\frac{\Delta_h N(x_i)}{2N(x_i)} - \frac{{}^{(3)}\mathcal{R}(x_i)}{9}\Big) + \mathcal{O}(t^2)\Big), \tag{2.18}$$

where ${}^{(3)}\mathcal{R}$ is the spatial Ricci curvature of Σ. When $N(x_i) = 1$, equation (2.18) reduces to the analogous formula given by Smerlak, as expected.

Taking the small time limit of (2.18) gives

$$\lim_{t\to 0}\langle d^2(x, x_i)\rangle_t = 6\kappa t N(x_i). \tag{2.19}$$

Recall that the lapse function converts between proper and coordinate time, which means that $N(x_i)\, t$ is the proper time at the origin of the Brownian motion. Thus (2.19) shows that the standard diffusion scaling law is recovered in the small time limit. This is to be expected since, by definition, the manifold on which the Brownian particle moves looks locally like Euclidean space. At later times the Brownian particle has had time to feel the effects of the nontrivial spacetime geometry and hence correction terms appear.

Furthermore, from (2.18) it can be seen how Ricci curvature and the Laplacian of the lapse function affect the motion of a Brownian particle. A *positive* Ricci curvature acts to *decrease* the MSD and hence *slows* the spreading of a Brownian particle. Conversely, a *negative* Ricci curvature acts to *speed up* the spreading of a Brownian particle. Prior to Smerlak’s observation in [10], this effect had already been noted in the biophysics literature in the context of diffusion on cell membranes in [12]. Since the lapse function is always positive, a positive Laplacian of the lapse function will speed up the spreading of a Brownian particle, whereas a negative Laplacian of the lapse function will slow the spreading of a Brownian particle. These effects will be further explored in chapter 3.

Chapter 3

Applications

This chapter discusses the physical implications of Smerlak's general relativistic diffusion equation in some specific astrophysical and cosmological models. The first section looks at Brownian motion in stars, while the second looks at Brownian motion in FLRW cosmological models. Unless otherwise stated, all calculations in this chapter are original to this report.

3.1 Diffusion in stars

In this section, Smerlak's diffusion equation (2.15) is used to study the behaviour of Brownian particles in general relativistic models of isolated stars. The standard way of modelling such stars using general relativity is to consider particular kinds of exact solutions of Einstein's field equations. These solutions are the so-called *static spherically symmetric perfect fluid* (SSSPF) solutions, which represent a static uniform sphere of *perfect fluid*. This means that the stellar fluid can be completely described by a radially varying density and a radially varying isotropic pressure. The condition of spherical symmetry means that every spacetime point lies on a two-surface which is a two sphere. A SSSPF solution models the interior of a star, but this needs to be matched to an exterior vacuum solution, which represents the spacetime surrounding the star. By Birkhoff's theorem, there is a unique exterior solution that should be matched to—this theorem says that the only spherically symmetric, asymptotically flat vacuum solution is the Schwarzschild exterior solution. The line element for the Schwarzschild exterior solution is given by

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1} dr^2 + r^2 d\Omega^2, \tag{3.1}$$

for $r > R$, where R is the radius of the star, M is mass of the star and $d\Omega = d\theta^2 + \sin^2(\theta)d\phi^2$.

There exist infinitely many SSSPF solutions, some more physically realistic than others. Since finding these solutions is straightforward, there exists a large amount of literature on the subject. Thus, in this report the primary concern will be to think about diffusion in physically realistic star models. A SSSPF solution will be classed as "physically realistic" if it satisfies certain conditions as specified by Delgaty and Lake [15]. Of the 127 candidate solutions considered by Delgaty and Lake, only nine satisfied all the conditions of physical applicability. Of these nine, the solutions considered here are the Tolman IV model and the

Goldman III model. The "unphysical" Schwarzschild constant density star is also considered, for reasons discussed below.

3.1.1 Schwarzschild constant density star

This solution is considered by Smerlak in [10] as an application of the curved-spacetime diffusion equation. Smerlak incorrectly assumes that the lapse function can be set equal to one at the centre of the star. The correct calculation is given below and the significance of the result is discussed.

The Schwarzschild constant density star was the first SSSPF solution found, published by Schwarzschild in 1916 [16]. The line element can be written in the form [11]

$$ds^2 = -\frac{1}{4}\Big(3\Big(1-\frac{2GM}{R}\Big)^{\frac{1}{2}} - \Big(1-\frac{2GMr^2}{R^3}\Big)^{\frac{1}{2}}\Big)^2 dt^2 + \Big(1-\frac{2GMr^2}{R^3}\Big)^{-1} dr^2 + r^2 d\Omega^2, \quad (3.2)$$

where G is Newton's gravitational constant, R is the star radius and $M = 4\pi\rho R^3/3$. Although this model is mathematically simple, it is physically unrealistic. It is unphysical since in order for the stellar matter to have constant density, it must be infinitely stiff, which means that the speed of sound in the fluid is infinite. Although this model fails to satisfy the physical condition of a subluminal sound speed, it is a useful starting point since it can be considered as an approximation to dense neutron stars, which have *nearly* uniform density [11].

For this geometry, it is found that the spatial Ricci curvature is constant, given by

$${}^{(3)}\mathcal{R} = \frac{12GM}{R^3}.$$

If our Brownian particle starts at the centre of the star, so that x_i is the spatial point $r = 0$, then

$$\Delta_h N(x_i) = \frac{3GM}{R^3}$$

and

$$N(x_i) = \frac{1}{2}\Big(3\Big(1-\frac{2GM}{R}\Big)^{\frac{1}{2}} - 1\Big),$$

which is clearly not equal to one. The value of $N(x_i)$ depends on the parameter $\beta = 2GM/R$. The construction of this stellar model constrains β to satisfy $0 < \beta < 8/9$. Notice that $N(x_i)$ becomes zero as β approaches the upper bound $8/9$. Physically, this says that the time-dilation at the origin, as witnessed by an observer that has proper time t, would become infinite as $\beta \to 8/9$.

Using (2.18), one finds that the MSD for a Brownian particle that starts in the centre of a constant density Schwarzschild star is

$$\langle d^2(x,x_i)\rangle_t = 6\kappa t N(x_i)\Big(1 + \underbrace{\kappa t N(x_i)\Big(\frac{3GM}{2N(x_i)R^3} - \frac{12GM}{9R^3}\Big)}_{\text{curvature correction term}} + \mathcal{O}(t^2)\Big). \quad (3.3)$$

Equation (3.3) is more general than the equation that Smerlak derives, which one obtains by setting $N(x_i) = 1$. That is, a rescaling of time in (3.3) does not give Smerlak's equation. This is because $N(x_i)$ appears nontrivially in the correction term in (3.3).

The behaviour of $N(x_i)$ means that, for physical time scales, the curvature correction term in the MSD can be significant. For example, the Brownian particle would reach the surface of the star when $\langle d^2(x, x_i)\rangle_t \approx R^2$. Using the second order expansion (3.3), one can calculate the magnitude of the coordinate time when this occurs. The magnitude of the correction term in (3.3) can then be examined when $\langle d^2(x, x_i)\rangle_t \approx R^2$ for different values of β; this is plotted in figure 3.1.

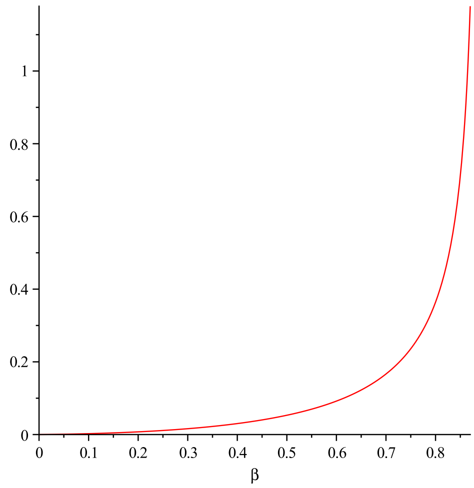


Figure 3.1: *Plot of the MSD curvature correction term for a Schwarzschild constant density star in terms of $\beta = 2GM/R$ for timescales on which $\langle d^2(x, x_i)\rangle \approx R^2$.*

As can be seen from figure 3.1, the correction term grows without bound as β approaches the critical value of 8/9. Thus, in the Schwarzschild constant density star, the curvature corrections to the motion of a Brownian particle will become significant for stars that have a large enough value of $2GM/R$. In particular, the effect of curvature is to increase the speed at which a Brownian particle spreads, and this spreading becomes faster for stars with a larger ratio of mass to radius. This conclusion is very different from Smerlak's, which was that the relative size of the curvature correction is insignificant (at most 1.3%).

As pointed out by Smerlak, the result that a Brownian particle spreads faster in a star seems paradoxical as we might expect that the attractive nature of a gravitational field would slow diffusion. However, as Smerlak points out [10]: "the infinitely frequent collisions between [the stellar medium] and the Brownian particle prevent the latter from falling down to the centre of the star".

To get a rough idea of the magnitude of this effect, suppose that we have a neutron star of constant density $\rho = 5.5 \times 10^{17}$ kgm^{-3} and a radius of 12 km, which is within the range of current neutron star models. The upper bound on β is equivalent to

$$R < \frac{c}{\sqrt{3\pi G\rho}} \approx 16 \text{ km},$$

so a radius of 12km is physically acceptable. Then

$$\beta = \frac{8\pi G R^2 \rho}{3c^2} \approx 0.5.$$

This corresponds to a correction term of approximately 5%.

3.1.2 Tolman IV solution

This solution, first published by Tolman [17], is described by the metric

$$ds^2 = -B^2\Big(1 + \frac{r^2}{A^2}\Big)dt^2 + \frac{1 + 2\frac{r^2}{A^2}}{(1 - \frac{r^2}{R^2})(1 + \frac{r^2}{A^2})}dr^2 + r^2 d\Omega^2.$$

The boundary of the star, r_b, is chosen such that the pressure vanishes there and is given by $3r_b^2 = R^2 - A^2$. Matching this solution with the Schwarzschild exterior vacuum solution gives $B^2 = A^2/R^2$. To satisfy certain physical conditions the parameter B must satisfy $\frac{2}{23} < B^2 < 1$. The Ricci curvature scalar at the centre of the star x_i is found to be

$$^{(3)}\mathcal{R}(x_i) = \frac{6(1 + B^2)}{R^2 B^2}.$$

Similarly, $N(x_i) = |B|$ and

$$\Delta_h N(x_i) = \frac{3}{|B|R^2}.$$

Thus, using (2.18), the MSD for a Brownian particle that starts at the centre of a Tolman IV star is given by

$$\langle d^2(x, x_i)\rangle_t = 6\kappa t|B|\Big(1 + \kappa t\frac{(1 - B^2)}{3|B|r_b^2}\Big(\frac{5}{6} - \frac{2B^2}{3}\Big) + \mathcal{O}(t^2)\Big).$$

For $|B|$ in the allowed range, the curvature correction term is always positive. Therefore, up to second order, curvature always causes the Brownian particle to spread faster than it would in a flat space. A plot of the curvature correction term for the allowed values of $|B|$ and for time scales when $\langle d^2(x, x_i)\rangle \approx r_b^2$ is given in figure 3.2.

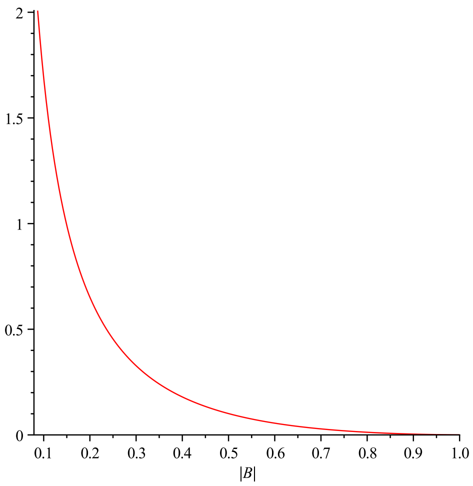


Figure 3.2: *Plot of the relative magnitude of the MSD curvature correction term for a Tolman IV model in terms of $|B|$ for timescales on which $\langle d^2(x, x_i)\rangle \approx r_b^2$.*

For the lowest allowed value of $|B|$, the correction term is approximately 2. That is, by the time the Brownian particle reaches the surface of the star, the effect of curvature is to make it spread three times as fast as it would in the absence of curvature effects. The dependence of the MSD correction term on $|B|$ is most likely because the radius of the star is larger for smaller $|B|$. That is, in a larger star the Brownian particle reaches the surface at later times and hence the correction term is bigger. The value of $|B|$ is also related to the pressure and density in the centre of the star, as given by Tolman [17]. A small value of $|B|$ implies that the star has a large central pressure and density. Thus the presence of a larger central pressure and density might also influence the corrections to the MSD.

3.1.3 Goldman III solution

This SSSPF solution was published by Goldman in 1978 [18] in a paper where he specifically seeks to find physically valid analytic solutions. The line element is given by

$$-A\Big(\frac{g(r)-1}{g(r)+1}\Big)dt^2 + B\Big(1+\frac{1}{g(r)}\Big)^2(dr^2 + r^2 d\Omega^2), \tag{3.4}$$

where $g(r) = \cosh(a + br^2)$, A, B, a, and b are constants that define the model, with a satisfying $a > 1.66$. Also, $b > 0$ ensures that the star has a finite radius and when b is larger the radius of the star is smaller. To satisfy one of the regularity conditions given by Delgaty and Lake, B must satisfy

$$B = \frac{(e^{2a}+1)^2}{(e^a+1)^4}.$$

Then, if x_i is the centre of the star, it is found that

$$\Delta_h N(x_i) = \frac{3\sqrt{A}b}{4B}\cosh^2(a)\mathrm{sech}^6(a/2)$$

and

$${}^{(3)}\mathcal{R}(x_i) = \frac{3b}{B}\cosh(a)\sinh(a)\mathrm{sech}^6(a/2).$$

Thus

$$\frac{\Delta_h N(x_i)}{2N(x_i)} - \frac{{}^{(3)}\mathcal{R}(x_i)}{9} = \frac{b\,\mathrm{sech}^6(a/2)\cosh(a)}{B}\Big(\frac{3}{8}\cosh(a)\coth(a/2) - \frac{1}{3}\sinh(a)\Big). \tag{3.5}$$

Due to the complexity of this model, there is no simple expression for the star radius in terms of the model parameters. This makes it difficult to plot the MSD correction term for time scales on which a Brownian particle reaches the surface of the star. Instead, a plot of the term given in (3.5), divided by b, is given in figure 3.3.

The maximum value of the correction term in equation (3.5), subject to the constraint that $a > 1.66$, is approximately $0.5b$. Thus, curvature effects in a Goldman III star will be significant when a is small and b is, at least, of order one. In particular, for all allowable values of the model parameters the spreading of a Brownian particle is faster relative to diffusion in Euclidean space.

In this model, the relationship between pressure, density and the model parameters is more complicated than in the Tolman IV model. However, there is a simple relationship given by

$$\cosh(a) = 1 + \frac{\rho_c}{3p_c},$$

where ρ_c and p_c are the values of density and pressure, respectively, at the centre of the star. Thus a larger a results in a larger value of ρ_c/p_c, which suggests that the combined effect of pressure and density influences the Brownian particle.

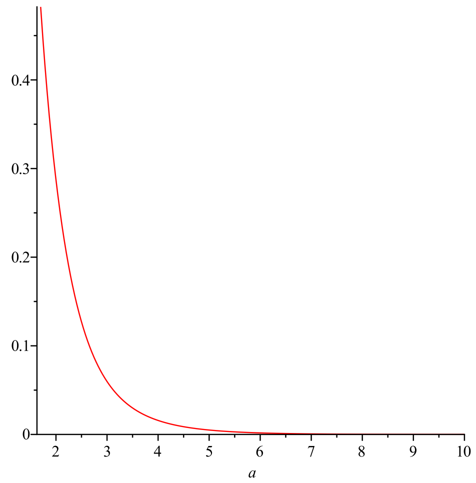


Figure 3.3: *Plot of the relative magnitude of the MSD curvature correction term (divided by b) for a Goldman III model in terms of the parameter a.*

3.2 Diffusion in a FLRW cosmology

In this section we look at Brownian motion in FLRW cosmological models. These models describe a homogeneous, isotropic expanding or contracting universe and underlie the standard model of cosmology.

The line element for a FLRW model can be written in terms of the cosmic scale factor $a(t)$ as

$$ds^2 = -dt^2 + a(t)^2\Big(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\Big). \tag{3.6}$$

The scalar $k \in \{-1, 0, 1\}$ determines the curvature of the spatial hypersurfaces Σ_t: $k = +1$ describes a closed hypersurface of positive curvature, $k = 0$ describes a flat hypersurface, and $k = -1$ describes an open hypersurface of negative curvature.

The spatial hypersurfaces Σ_t are changing with time according to the scale factor $a(t)$, which means that the Laplace-Beltrami operator also changes with time. Let Δ_t denote the Laplace-Beltrami operator and h_t^{ij} the inverse of the intrinsic metric on Σ_t. The relationship between Δ_t and Δ_{t_i} can be found be considering these operators in a coordinate basis:

$$\begin{aligned}
\Delta_t(\cdot) &= \frac{1}{\sqrt{|h_t|}}\partial_r\Big(\sqrt{|h_t|}h_t^{rr}\partial_r(\cdot)\Big)\\
&= \frac{1}{a^2(t)}\frac{1}{\sqrt{|h_{t_i}|}}\partial_r\Big(\sqrt{|h_{t_i}|}h_{t_i}^{rr}\partial_r(\cdot)\Big)\\
&= \frac{1}{a^2(t)}\Delta_{t_i}(\cdot),
\end{aligned} \tag{3.7}$$

where spherical symmetry was used to imply that only the ∂_r derivative contributes. The initial value of the scale factor, $a(t_i)$, has also been set to one without loss of generality.

3.2.1 Solving the diffusion equation in a FLRW universe

The lapse function for a FLRW model is given by $N = 1$ and the expansion is $\theta = 3\dot{a}(t)/a(t)$. Thus, the general relativistic diffusion equation (2.15) for a FLRW spacetime is

$$\partial_t p + 3\frac{\dot{a}}{a}p = \kappa\Delta_t(p).$$

The left-hand side can be rewritten in terms of a single derivative, giving

$$\partial_t(pa^3) = \kappa\Delta_t(pa^3). \tag{3.8}$$

Now, using (3.7), this becomes

$$\partial_t(pa^3) = \frac{1}{a^2(t)}\kappa\Delta_{t_i}(pa^3). \tag{3.9}$$

Hence, the diffusion equation in a FLRW universe (3.9) is the static equation (2.16) with the replacement $p \to pa^3$ and $\kappa \to \kappa/a^2$.

The diffusivity κ is the proper diffusivity as measured by an observer comoving with the expansion of space. In general, one would expect the proper diffusivity to change in time due to the diluting effect of the expansion of space. Since κ is a phenomenological quantity, its value is usually determined empirically for any given material. A particular form will be assumed for κ when considering specific models later in this section, but for now the dependence is kept arbitrary. Thus, the equation to be solved for the heat kernel solution $K_t(x, x_i)$ is

$$\partial_t(K_t a^3) = \frac{1}{a^2(t)}\kappa(t)\Delta_{t_i}(K_t a^3), \tag{3.10}$$

with the initial condition

$$\lim_{t\to t_i} K_t(x, x_i) = \delta_{t_i}(x, x_i),$$

where δ_{t_i} is the Dirac delta distribution on Σ_{t_i}. Equation (3.10) is solved in a manner similar to solving the Schrödinger equation with a time dependent Hamiltonian in quantum mechanical problems. The solution is

$$K_t(x, x_i)a^3(t) = \exp\Big(\int_{t_i}^{t} \frac{1}{a^2(t')}\kappa(t')\Delta_{t_i}\, dt'\Big)\delta_{t_i}(x, x_i).$$

Thus the heat kernel is given by

$$K_t(x, x_i) = \frac{1}{a^3(t)}\exp(\bar{t}\Delta_{t_i})\delta_{t_i}(x, x_i),$$

where

$$\bar{t} = \int_{t_i}^{t} \frac{\kappa(t')}{a^2(t')}\, dt'. \tag{3.11}$$

3.2.2 The MSD in a FLRW universe

Finding the MSD on Σ_t is superficially similar to the static case that was considered in section 2.3.4, except that now one has to be careful to account for the dynamics of the spatial hypersurfaces.

If $d_t^2(x, x_i)$ denotes the square of the geodesic distance between the points x and x_i on Σ_t, then the MSD is given by

$$\langle d_t^2(x, x_i)\rangle_t = \frac{1}{a^3(t)} \int_{\Sigma_t} \exp(\bar{t}\Delta_{t_i})\delta_{t_i}(x, x_i) d_t^2(x, x_i)\, dV_t(x). \tag{3.12}$$

Equation (3.12) cannot be evaluated straight away. This is because it contains the operator Δ_{t_i}, which strictly acts on functions that are defined on the hypersurface Σ_{t_i}, acting on the function d_t^2, which is defined on the hypersurface Σ_t. Additionally, (3.12) involves integrating the Dirac delta distribution δ_{t_i}, which is strictly defined on the hypersurface Σ_{t_i}, over the hypersurface Σ_t. To remedy the first problem, simply use the transformation of the Laplace-Beltrami operator that was deduced in (3.7). To fix the second problem, note that

$$1 = \int_{\Sigma_t} \delta_t(x, x_i) dV_t(x) = \int_{\Sigma_{t_i}} \delta_t(x, x_i) a^3(t) dV_{t_i}(x), \tag{3.13}$$

where the fact that a proper volume element changes according to $a^3(t)$ was used to change the integration measure. Equation (3.13) implies that in an expanding universe the Dirac delta distribution transforms according to

$$\delta_t(x, x_i) = \frac{1}{a^3(t)} \delta_{t_i}(x, x_i).$$

The MSD can now be written in terms of mathematical objects that are defined on Σ_t,

$$\langle d_t^2(x, x_i)\rangle_t = \int_{\Sigma_t} \exp(\bar{t}a^2(t)\Delta_t)\delta_t(x, x_i) d_t^2(x, x_i)\, dV_t(x). \tag{3.14}$$

Let $\langle A, \phi\rangle_t$ define the pairing between a distribution A and a test function ϕ in Σ_t; that is,

$$\langle A, \phi\rangle_t = \int_{\Sigma_t} A(\phi)\, dV_t.$$

To evaluate equation (3.14), one needs to consider terms of the form

$$\langle \Delta_t \delta_t, d_t^2\rangle_t = \langle \delta_t, \Delta_t d_t^2\rangle_t = \Delta_t d_t^2(x_i, x_i). \tag{3.15}$$

As was noted in appendix A, to evaluate terms such as (3.15), one has to use the geometric relations $\Delta_h d^2(x_i, x_i) = 2D$ and $\Delta_h^2 d^2(x_i, x_i) = -4\,{}^{(D)}\mathcal{R}(x_i)/3$. Thus, the small time asymptotic expansion of the MSD in a FLRW universe is given by

$$\begin{aligned}
\langle d_t^2(x, x_i)\rangle_t &= \langle \exp(\bar{t}a^2\Delta_t)\delta_t, d_t^2\rangle \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \langle (\bar{t}a^2)^n \Delta_t^n \delta_t, d_t^2\rangle \\
&= d_t^2(x_i, x_i) + \langle \bar{t}a^2\Delta_t\delta_t, d_t^2\rangle + \frac{1}{2}\langle (\bar{t}a^2)^2\Delta_t^2\delta_t, d_t^2\rangle + \dots \\
&= \bar{t}a^2\Delta_t d_t^2(x_i, x_i) + \frac{1}{2}(\bar{t}a^2)^2\Delta_t^2 d_t^2(x_i, x_i) + \mathcal{O}((\bar{t}a^2)^3) \\
&= 2D\bar{t}a^2 - 2(\bar{t}a^2)^2 \frac{{}^{(D)}\mathcal{R}(x_i)}{3} + \mathcal{O}((\bar{t}a^2)^3).
\end{aligned}$$

In a 3-dimensional Friedmann universe, the quantity $\Delta_t^3 \delta_t(x, x_i)$ vanishes. Also, in 3-dimensions the spatial Ricci curvature on a spatial hypersurface Σ_t is found to be

$$^{(3)}\mathcal{R} = \frac{6k}{a(t)^2}. \tag{3.16}$$

This is independent of position, as one would expect due to the assumed homogeneity of space. Hence, in 3-dimensions the asymptotic expansion of the MSD in a FLRW universe is

$$\langle d_t^2(x, x_i)\rangle_t = 6\bar{t}a^2\Big(1 - \frac{2\bar{t}k}{3} + \mathcal{O}((\bar{t}a^2)^3)\Big), \tag{3.17}$$

where

$$\bar{t} = \int_{t_i}^{t} \frac{\kappa(t')}{a^2(t')}\, dt'.$$

For equation (3.17) to be a well-defined asymptotic expansion, the function $\bar{t}a^2$ needs to tend to zero as t tends to t_i. Clearly, by definition (3.11), $\bar{t} \to 0$ as $t \to t_i$. Since the cosmic scale factor $a(t)$ measures the *relative* expansion of space, one is always free to define $a(t_i) = 1$. This shows that equation (3.17) is a well-defined asymptotic expansion.

When $\Delta t = t - t_i$ is small, one can write

$$\bar{t} = \int_{t_i}^{t_i+\Delta t} \frac{\kappa(t')}{a^2(t')}\, dt' \approx \Delta t \frac{\kappa(t_i)}{a^2(t_i)}.$$

Thus, the $t \to t_i$ limit of equation (3.17) is given by

$$\lim_{t\to t_i} \langle d_t^2(x, x_i)\rangle_t = \lim_{t\to t_i} \Big(6a^2(t) \int_{t_i}^{t} \frac{\kappa(t')}{a^2(t')}\, dt'\Big) = 6\kappa(t_i)(t - t_i),$$

which is the MSD in a Euclidean space, as expected.

3.2.3 Spatially flat FLRW models

Consider a spatially flat FLRW universe with zero cosmological constant that is filled with a single-species fluid. It is assumed that the fluid satisfies an equation of state of the form $P = w\rho$, where P is pressure and ρ is density. Then Einstein's field equations imply that

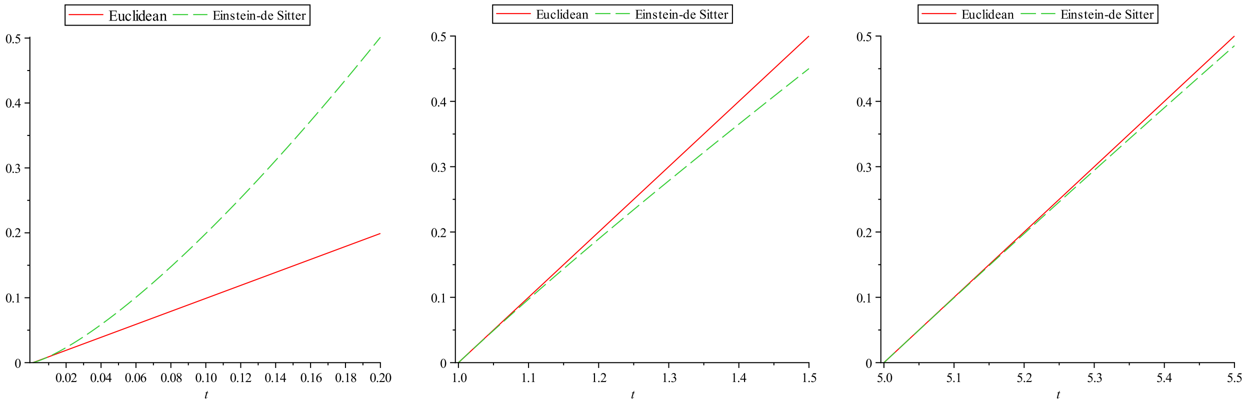
$$a(t) = \Big(\frac{t}{t_i}\Big)^{\frac{2}{3(1+w)}}, \tag{3.18}$$

where $a(t_i)$ has been set to one and $-1 < w \leq 1$.

To do calculations, a particular form of $\kappa(t)$ must be assumed. Since κ is intrinsic to the fluid substrate, which gets diluted as the universe expands, it is reasonable to assume that $\kappa(t)$ is proportional to some power of the scale factor $a(t)$; hence it will be assumed that $\kappa(t) = \kappa_i a^m(t)$ for some m, where κ_i is the diffusivity at time t_i.

Consider now a universe with $a \propto t^n$ and $\kappa \propto a^m$. These conditions will be satisfied in spatially flat universes filled with a single-species fluid, in which case one has $n > 1/3$. These conditions also describe the late time behaviour of many non-flat models. With these assumptions, the first term in equation (3.17) is

$$6\bar{t}a^2(t) = \frac{\kappa_i(t^{nm-2n+1}t_i^{2n-nm} - t_i)}{nm - 2n + 1}. \tag{3.19}$$



(a) t_i= *0.001 second* (b) t_i= *1 second* (c) t_i= *5 second*

Figure 3.4: *Comparison of the MSD for a Brownian particle that starts at t_i in a Euclidean space and an Einstein-de Sitter space.*

Expanding (3.19) using series expansions gives

$$\lim_{t_i\to\infty} 6\bar{t}a^2(t) = 6\kappa_i(t-t_i) + \frac{3\kappa_i(t-t_i)^2(2+m)n}{t_i} + \mathcal{O}((t-t_i)^2/t_i^2). \tag{3.20}$$

Equation (3.20) shows that for all flat universes with $a \propto t^n$ and $\kappa \propto a^m$, the MSD of a Brownian particle tends to the MSD in a Euclidean space as time increases. Furthermore, if $(2+m)n < 0$, then the MSD in an expanding universe approaches the Euclidean MSD function from below; if $(2+m)n > 0$, then the MSD in an expanding universe approaches the Euclidean MSD function from above. If $m = 2$, expanding (3.19) to third order shows that the MSD approaches the Euclidean MSD from above.

To give this analysis some concreteness, consider the Einstein-de Sitter universe. This is a spatially flat FLRW universe filled with zero-pressure matter, which means that $w = 0$, $a \propto t^{2/3}$, and $\rho \propto a^{-3}$. It will be assumed that the diffusivity is proportional to the density, which gives $\kappa = \kappa_i a^{-3}(t)$. Using (3.17), the MSD in an Einstein-de Sitter universe is

$$\langle d_t^2(x,x_i)\rangle = \frac{18\kappa_i t^{4/3}}{7t_i^{1/3}}\left(1-\frac{t_i^{7/3}}{t^{7/3}}\right) + \mathcal{O}((\bar{t}a^2)^4). \tag{3.21}$$

When $(t-t_i)/t_i$ is small, (3.21) can be expanded in series as

$$\langle d_t^2(x,x_i)\rangle_t = 6\kappa_i(t-t_i)\left(1-\frac{(t-t_i)}{3t_i} + \mathcal{O}\left(\frac{(t-t_i)^2}{t_i^2}\right)\right). \tag{3.22}$$

Hence expansion causes the Brownian particle to spread slower in an Einstein-de Sitter universe than in a flat Euclidean space.

Figure 3.4 compares $\langle d^2(x,x_i)\rangle/6\kappa_i$ at times $t_i = 0.001$s, $t_i = 1$s and $t_i = 5$s between the Euclidean space and an Einstein-de Sitter universe. In these plots, the MSD has been plotted for time intervals on which the asymptotic expansion holds—that is, when $a^2(t)\bar{t}$ is small. For larger values of t_i, the graphed difference between the Euclidean and Einstein-de Sitter universe MSDs becomes undetectable, as expected from (3.22).

Even with a non-zero cosmological constant, our universe is matter-dominated for a certain range of cosmic times. In these epochs, spatial curvature can be neglected, so equation

(3.22) is a good approximation to the MSD for a significant fraction of the history of the universe. These calculations suggest that the effects of expansion on the motion of a Brownian particle are undetectable during these epochs.

To investigate the effects of general relativity on Brownian motion in the early universe, it is useful to consider a spatially flat de Sitter universe. In this model there is a non-zero cosmological constant, Λ, and exponential expansion. The scale factor is given by $a(t) = e^{(t-t_i)/\alpha}$, where the constant α is small and is related to Λ by $\alpha = \sqrt{3/\Lambda}$. Such expansion is a good approximation to inflation models that have a classical de Sitter phase, such as "slow roll inflation" models.

For the de Sitter model, the asymptotic expansion of the MSD is given by

$$\langle d_t^2(x, x_i)\rangle_t = \frac{6\kappa\alpha e^{-m(t-t_i)/\alpha}}{2-m}\Big(e^{(t-t_i)(2-m)/\alpha} - 1\Big) + \mathcal{O}((\bar{t}a^2)^4, \tag{3.23}$$

where $\kappa(t) = \kappa_i a^m(t)$. Taking $t - t_i$ to be small, a series expansion of (3.23) gives

$$\langle d_t^2(x, x_i)\rangle_t = 6\kappa_i(t - t_i)\Big(1 + (t - t_i)\frac{(2-m)}{2\alpha} + \mathcal{O}((t - t_i)^2)\Big). \tag{3.24}$$

Equation (3.24) implies that general relativistic effects will significantly affect the motion of a Brownian particle during inflationary expansion and that m cannot be greater than 2, since this would give a negative MSD.

Consider an inflationary period from $t_i = 10^{-36}$s to $t_f = 10^{-32}$s and let the scale factor increase by a factor of e^{60} over this interval—this corresponds to the volume of the universe increasing by a factor of 10^{78}. Then $e^{(t_i-t_f)/\alpha} = e^{-60}$ implies that $\alpha = 1.7 \times 10^{-34}$, since α is assumed to be constant over this interval. This implies that the ratio of the mean distance travelled by a Brownian particle in a de Sitter space to that in a Euclidean space is given by

$$\frac{\sqrt{\langle d_t^2(x, x_i)\rangle_t}_{\text{de Sitter}}}{\sqrt{\langle d_t^2(x, x_i)\rangle_t}_{\text{Euclidean}}} = \sqrt{1 + 30(2 - m)}.$$

So if $\kappa \propto a^{-3}(t)$, a Brownian particle diffusing during this de Sitter phase of inflation will travel on average $\sqrt{150} \approx 12$ times further than it would in a Euclidean space over the same time period. This is an ad hoc calculation, which in view of the small timescales may be a regime in which the infinite speed of light approximation is not to be trusted. However, it shows that in principle inflationary expansion can have significant effects on Brownian motion.

Chapter 4

Discussion

Formulating general relativistic versions of classical theories has in many instances proven to be a difficult task. In particular, there is no known completely satisfactory general relativistic theory of diffusion and, a fortiori, Brownian motion. In this report, we considered a curved-spacetime diffusion equation recently proposed by Smerlak. This equation is the limit of a fully general relativistic diffusion equation in which the speed of light goes to infinity. In other words, the difference between Smerlak's diffusion equation and the classical diffusion equation is the inclusion in the former of a nontrivial spacetime geometry. As such, this equation is valid in the regime where the time scales of diffusion are larger than the relaxation time of the background substrate. This is the same regime in which Eckart's dissipative relativistic hydrodynamics applies.

One might expect that a Brownian particle that moves in a curved spacetime will feel the effects of the nontrivial geometry of the manifold in which it lives. Smerlak's equation confirms that this is the case. To quantify this effect, one can consider the mean squared displacement (MSD) of such a Brownian particle. In particular, the departure of the MSD from the diffusion scaling law, which says that the MSD is proportional to the square root of time, measures the significance of spacetime curvature on the motion of a Brownian particle.

Smerlak [10] considered the motion of a Brownian particle in a Schwarzschild constant density star as an astrophysical application of his diffusion equation. We found a mistake in his reasoning, which is corrected in this report. We found that the effects of curvature on a Brownian particle are significant in certain constant density stars, in contrast to the conclusion reached by Smerlak. Specifically, the MSD is always larger than in a Euclidean space and can be made arbitrarily large by considering constant density stars that are close to black hole collapse. Using a radius and mean density compatible with current neutron star models, the correction term was found to be typically 5% on the timescales it takes an average Brownian particle to move from the centre of a neutron star to the surface. To further investigate this finding, we considered Brownian motion using other general relativistic models of stars that are more physically realistic than the Schwarzschild constant density star. In particular, we considered the Tolman IV model and the Goldman III models, both of which belong to the small set of static spherically symmetric perfect fluid solutions that satisfy certain physically reasonable conditions that were proposed by Delgaty and Lake [15].

It was found that the MSD of a Brownian particle that starts at the centre of a Tolman IV star is larger than it would be in a Euclidean space; on physical time scales, this effect can be up to three times as large as it would be in the absence of curvature. However, the exact size of the correction terms depends on the value of a model parameter. Similarly, in the Goldman III model we found that the effects of curvature could increase the MSD by a significant amount, depending on the value of the model parameters.

The astrophysical situations considered in this report and by Smerlak relied on the simplifying condition of a static spacetime geometry. Although Smerlak's diffusion equation is suitable for more general spacetime structures, he derives a small-time asymptotic expansion of the MSD only for static geometries. We explicitly investigated the extension of Smerlak's MSD relation to cosmological models in which the universe is expanding and hence non-static. Specifically, we derived the small-time asymptotic expansion of the MSD for a general FLRW model. The time-dependence of the spatial hypersurfaces, and the mathematical objects defined therein, meant that this derivation required several significant modifications to the analogous derivation for static spacetimes.

The MSD formula for Brownian motion in a FLRW universe that was derived in this report, equation (3.17), allows for an arbitrary time dependence in the proper diffusivity. It is natural to assume that the diffusivity is proportional to some power of the scale factor. With this nonrestrictive assumption, we found that the class of zero-curvature FLRW solutions that have $a \propto t^n$ yield an MSD that approaches the Euclidean MSD at late times. In other words, the effects of expansion in a zero-curvature universe that expands according to $a \propto t^n$ are undetectable for large enough times. By considering a de Sitter phase of expansion for early times, as proposed by some models of inflation, we found that general relativistic effects are, in principle, important in the very early inflationary universe.

The work in this report represents an evaluation and modest extension of Smerlak's work on Brownian motion in curved spacetimes. There are several directions for future work using Smerlak's curved-spacetime diffusion equation in cosmological and astrophysical situations that would be worth pursuing. Firstly, it is of interest to investigate the effects of general relativity on Brownian motion near more dense astrophysical objects, such as in accretion disks near a Kerr black hole. In addition, if curvature effects are significant for the motion of a Brownian particle in more generic relativistic star models, then one should investigate the observational consequences. Further investigation into the effects of general relativity on early universe diffusion processes should also be done. For example, the FLRW MSD solution found here could give insight into the effects of general relativity on the photon diffusion damping (Silk damping) that occurred during recombination. Such future work could prove useful in yielding insights into a fully general relativistic theory of Brownian motion, an as yet unfulfilled unification of two of Einstein's great contributions to physics.

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Appendix A

MSD calculation

In this appendix, the general second order expansion of the mean squared displacement for a Brownian particle in a $(D+1)$-dimensional static spacetime is given.

$$\begin{aligned}
\langle d^2(x, x_i)\rangle_t &= \sum_{n=0}^{\infty} \frac{(\kappa t)^n}{n!} \big((N\Delta_h)^n d^2\big)(x_i, x_i) \\
&= d^2(x_i, x_i) + \kappa t(N\Delta_h d^2)(x_i, x_i) + \frac{(\kappa t)^2}{2}((N\Delta_h)(N\Delta_h)d^2)(x_i, x_i) + \mathcal{O}(t^3) \\
&= \kappa t(N\Delta_h d^2)(x_i, x_i) + \frac{(\kappa t)^2}{2}(N^2\Delta_h^2 d^2 + N(\Delta_h N)(\Delta_h d^2))(x_i, x_i) + \mathcal{O}(t^3) \\
&= \kappa t N(x_i)(\Delta_h d^2)(x_i, x_i) + \frac{(\kappa t)^2}{2}\Big(N(x_i)^2\Delta_h^2 d^2(x_i, x_i) + N(x_i)\Delta_h N(x_i)\Delta_h d^2(x_i, x_i)\Big) \\
&\quad + \mathcal{O}(t^3) \\
&= 2D\kappa t N(x_i)\Big(1 + \kappa t N(x_i)\Big(\frac{\Delta_h N(x_i)}{2N(x_i)} - \frac{{}^{(D)}\mathcal{R}(x_i)}{3D}\Big) + \mathcal{O}(t^2)\Big). \qquad \text{(A.1)}
\end{aligned}$$

To perform the last step of this calculation, one has to use the geometric relations $\Delta_h d^2(x_i, x_i) = 2D$ and $\Delta_h^2 d^2(x_i, x_i) = -4{}^{(D)}\mathcal{R}(x_i)/3$, which are listed in, for example, [19].